Journal of Applied Mechanics and Technical Physics, Vol. 42, No. 3, pp. 546–550, 2001

ROLLING FRICTION AS A VISCOELASTIC DISSIPATIVE PROCESS

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UDC 531.45+539.374

The viscous additions to the stress tensor on the half-space surface over which a ball moves are calculated with the use of a viscosity (dissipative) tensor. The rolling friction force which corresponds to the Coulomb law and which is proportional to the velocity and which is simultaneously the lower estimate for the sliding friction force is found. Expressions for the radial and vertical displacements on the surface of an elastic half-space are given.

The rolling friction can be considered as a result of the joint action of various dissipative processes inside a rolling body and inside a body over which the motion occurs. These processes can be excitation of acoustic waves, thermal conduction upon nonuniform heating, and dissipative processes connected with the finite strain rate, i.e., internal friction (viscosity). Below, the processes caused by internal friction are considered.

If the strain rate is low and, therefore, the energy dissipation is insignificant (which occurs upon rolling under normal conditions), the effects of viscosity upon deformation can be described by the viscosity or dissipative tensor η_{iklm} [1]. Using this tensor, one can calculate the dissipative-power volume density, the internal friction forces in a deformable body, and the additions to the stress tensor caused by viscosity. In an isotropic body, the tensor η_{iklm} has only two independent components: the viscosity coefficient upon pure shift η and the viscosity coefficient upon triaxial compression ζ. In this case, the viscous stress-tensor component is determined by the expression [1]

$$\sigma'_{ik} = 2\eta(\dot{u}_{ik} - \dot{u}_{ll}\delta_{ik}/3) + \zeta\dot{u}_{ll}\delta_{ik}, \tag{1}$$

where $\dot{u}_{ik}$ are the time derivatives of the strain tensor and δ_{ik} is the Kronecker symbol; summation is performed over repeated indices: $u_{ll} = u_{xx} + u_{yy} + u_{zz} = \operatorname{div} \boldsymbol{u}$ ($\boldsymbol{u}$ is the strain vector).

Upon rolling of a body along a certain surface, a stress state whose elastic part is symmetric along the motion direction and, hence, does not exert an effect on the rolling arises. The viscous stress component on the rolling surface is nonsymmetric along the motion direction, because the signs of the strain rates in front of and behind the rolling body are different (compression occurs in front of the body, and unloading occurs behind this body). For this reason, the pressures across the leading and trailing parts of the body are different, which causes rolling resistance, i.e., rolling friction of the body.

As an example, we consider the rolling of a ball along the flat surface of a viscoelastic half-space. As the viscous corrections to the strain state are small in this case, we consider that the contact spot is a circle of radius a determined by the solution of a corresponding contact problem (the Hertz problem [1, 2]) and one can use the strain tensor obtained from the solution of a related elastic problem to determine σ'_{ik}. Therefore, the elastic part of the strain is axisymmetric and the strain vector $\boldsymbol{u}$ has only two components $u_\rho(\rho, z)$ and $u_z(\rho, z)$ in the cylindrical coordinate system with the z axis passing through the center of the ball inside the half-space along whose surface it rolls. Here

$$u_{ll} = \operatorname{div} \boldsymbol{u} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho u_\rho) + u_{zz},$$

where $u_{zz} = \partial u_z/\partial z$ and $\rho = \sqrt{x^2 + y^2}$. Thus, according to (1), the viscous-stress tensor component we need hereinafter has the form

Moscow State Technological University “Stankin,” Moscow 101402. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 42, No. 3, pp. 197–202, May–June, 2001. Original article submitted February 15, 1999; revision submitted September 25, 2000.

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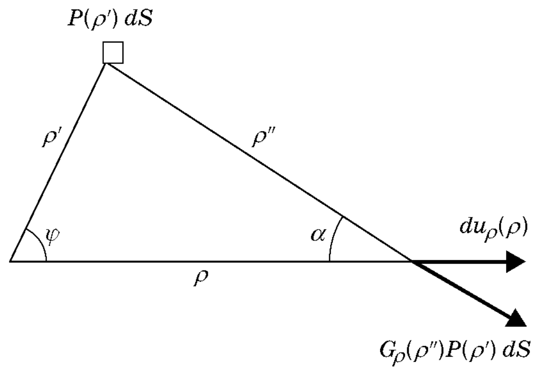

Fig. 1

$$\sigma'_{zz}(\rho,z)=\Big(\zeta-\frac{2}{3}\,\eta\Big)\frac{1}{\rho}\,\frac{\partial}{\partial\rho}\,(\rho\dot{u}_\rho)+\Big(\frac{4}{3}\,\eta+\zeta\Big)\dot{u}_{zz}. \tag{2}$$

To find the components of the elastic strain-vector part, we use the Green functions, i.e., the responses to the forces which for the case where the forces are perpendicular to the half-space surface take the form [1, 2]

$$G_\rho(\rho,z)=\frac{1+\nu}{2\pi E}\Big[\frac{z}{r^3}-\frac{1-2\nu}{r(r+z)}\Big]\rho,\qquad G_z(\rho,z)=\frac{1+\nu}{2\pi E}\Big[\frac{2(1-\nu)}{r}+\frac{z^2}{r^3}\Big], \tag{3}$$

where E, ν are Young's modulus and Poisson's ratio for the half-space and $r=\sqrt{\rho^2+z^2}$. The Green functions allow one to find the components of the strain vector $\boldsymbol{u}$ with the use of the known pressure $P(x,y)$ on the surface of the deformable half-space, which is presented in the form

$$P(x,y)=\frac{3N}{2\pi a^2}\sqrt{1-\Big(\frac{\rho}{a}\Big)^2}\qquad(\rho\leqslant a) \tag{4}$$

in the contact region in the case of impression into the surface of the ball half-space according to [1, 2]. Here N is the normal-pressure force of the ball and $a=\{(3/4)[(1-\nu^2)/E+(1-\nu_{\rm b}^2)/E_{\rm b}]RN\}^{1/3}$ is the radius of the contact spot (R is the ball radius) and $E_{\rm b}$ and $\nu_{\rm b}$ are Young's modulus and Poisson's ratio of the ball, respectively).

We obtain an expression for u_ρ on the half-space surface. With allowance for (3), one can write (Fig. 1)

$$du_\rho=G_\rho(\rho'',0)\cos\alpha P(\rho')\,dS=-\frac{(1+\nu)(1-2\nu)}{2\pi E}\,\frac{\rho-\rho'\cos\varphi}{\rho'^2+\rho^2-2\rho'\rho\cos\varphi}\,P(\rho')\rho'\,d\rho'\,d\varphi,$$

whence

$$u_\rho(\rho,0)=-\frac{(1+\nu)(1-2\nu)}{\pi E}\int\limits_0^a d\rho'\rho'P(\rho')\int\limits_0^\pi\frac{\rho-\rho'\cos\varphi}{\rho'^2+\rho^2-2\rho'\rho\cos\varphi}\,d\varphi.$$

One can show (see [3, formula 3.613.2]) that

$$\int\limits_0^\pi\frac{\rho-\rho'\cos\varphi}{\rho'^2+\rho^2-2\rho'\rho\cos\varphi}\,d\varphi=\pi\,\frac{H(\rho-\rho')}{\rho},$$

where $H(\rho)$ is the Heaviside (unit) function. Finally, we obtain

$$u_\rho(\rho,0)=-\frac{(1+\nu)(1-2\nu)}{E\rho}\int\limits_0^\rho P(\rho')\rho'\,d\rho'. \tag{5}$$

It is noteworthy that expression (5) for the radial strain on the half-space surface under axisymmetric normal loading is simpler and more convenient for calculations than that given in [2].

We obtain an expression for $u_{zz}=\partial u_z/\partial z$ on the boundary surface which is necessary to calculate the viscous-stress tensor (2). The strain-vector component u_z is a convolution of the Green function G_z and the surface pressure P [1]

$$u_z(x,y,z)=\iint G_z(x-x',y-x',z)P(x',y')\,dx'\,dy',$$

which takes the following form at the axisymmetric pressure $P(x,y) \equiv P(\rho)$:

$$\bar{u}_z(w,z) = 2\pi \bar{G}_z(w,z)\bar{P}(w). \tag{6}$$

Here $\bar{u}_z$, $\bar{G}_z$, and $\bar{P}$ is the zero-order Hankel transformations of the corresponding functions [4]. From (3), one can obtain (see [3, formulas 6.554.1 and 6.554.4])

$$\bar{G}_z(w,z) = \int\limits_0^\infty G_z(\rho,z)J_0(w\rho)\rho\,d\rho = \frac{1+\nu}{2\pi E}\Big(\frac{2(1-\nu)}{w} + z\Big)\mathrm{e}^{-wz}. \tag{7}$$

Here $J_0(x)$ is the Bessel function. Substituting expression (7) into (6) and differentiating (6) with respect to z to find $\bar{u}_{zz}$, it is easy to show that since $\partial\bar{G}_z/\partial z$ for $z=0$ is a constant which does not depend on the parameter of the Hankel transformation w, the Hankel transformations $\bar{u}_{zz}$ and $\bar{P}$ are proportional for $z=0$ and, consequently, the functions themselves are proportional. Thus, on the half-space surface, we have

$$u_{zz}(\rho,0) = -\frac{(1+\nu)(1-2\nu)}{E}\,P(\rho). \tag{8}$$

Substituting (5) and (8) into (2), we obtain the expression for the viscous-stress tensor on the surface

$$\sigma'_{zz} = -\frac{2(1+\nu)(1-2\nu)}{E}\Big(\zeta + \frac{1}{3}\,\eta\Big)\dot{P}(\rho); \tag{9}$$

note that if the elastic stresses $P(\rho)$ created by the rolling ball displace with velocity v in the positive direction of the x axis, then

$$\dot{P}(\rho) = \frac{dP}{d\rho}\,\dot{\rho} = -v\,\frac{x}{\rho}\,\frac{dP}{d\rho} = -v\,\frac{3N}{2\pi a^4}\,\frac{x}{\sqrt{1-(x^2+y^2)/a^2}} \tag{10}$$

with allowance for (4). It is clear that the coefficient at $\dot{P}(\rho)$ in (9) has the dimension of time and, as one can show, it is the relaxation time of the strain-tensor components on the surface of a viscoelastic body, i.e., the relaxation time of the surface relative strains. Introducing the notation for the relaxation time on the surface

$$\tau = \frac{2(1+\nu)(1-2\nu)}{E}\Big(\zeta + \frac{1}{3}\,\eta\Big) = \frac{\zeta + \eta/3}{K + \mu/3}, \tag{11}$$

where $K = E/[3(1-2\nu)]$ is the modulus of triaxial compression and $\mu = E/[2(1+\nu)]$ is the shear modulus, with allowance for expressions (10) we write expression (9) for the viscous-stress tensor on the surface in the form

$$\sigma'_{zz}(x,y) = -v\tau\,\frac{3N}{2\pi a^4}\,\frac{x}{\sqrt{1-(x^2+y^2)/a^2}}.$$

The moment of rolling friction forces can be obtained by integration of the surface pressure equal to the viscous-stress tensor with the opposite sign over the area of a contact spot of radius a:

$$M_{\mathrm{fr}} = -\int\limits_{-a}^{a} dx\,x \int\limits_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sigma'_{zz}(x,y)\,dy = v\tau N\,\frac{3}{2\pi}\int\limits_{-1}^{1} d\theta\theta^2 \int\limits_{-1}^{1} \frac{d\vartheta}{\sqrt{1-\vartheta^2}} = v\tau N. \tag{12}$$

Considering that the rolling friction force F_{fr} is applied to the center of the rolling ball, we find from (12) that

$$F_{\mathrm{fr}} = N\,\frac{\tau v}{R}, \tag{13}$$

where R is the radius of the rolling ball, N is the normal-pressure force, v is the rolling velocity, and τ is the relaxation time on the rolling surface determined in (11). The proportionality of the friction force (13) to the ratio N/R corresponds to the Coulomb law.

It is necessary to note that upon sliding of the ball over an ideally slippery surface, when the tangent stresses caused by sliding friction are absent, the stress state remains the same as upon rolling of a ball, i.e., the sliding friction force is different from zero upon sliding over an ideally slippery surface. Thus, the force value determined from (13) gives the lower estimate of the sliding friction force of the ball.

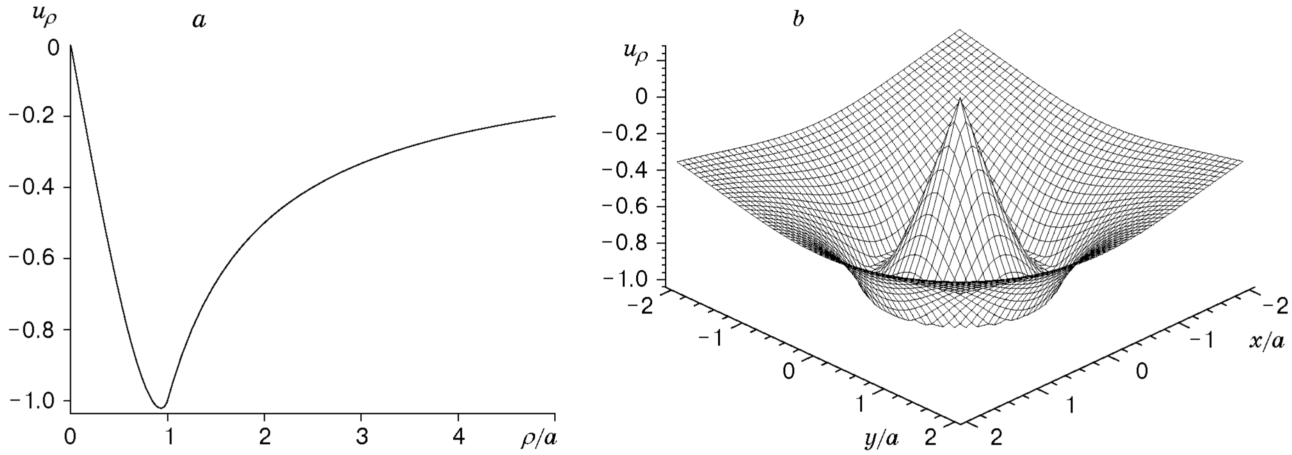

Fig. 2

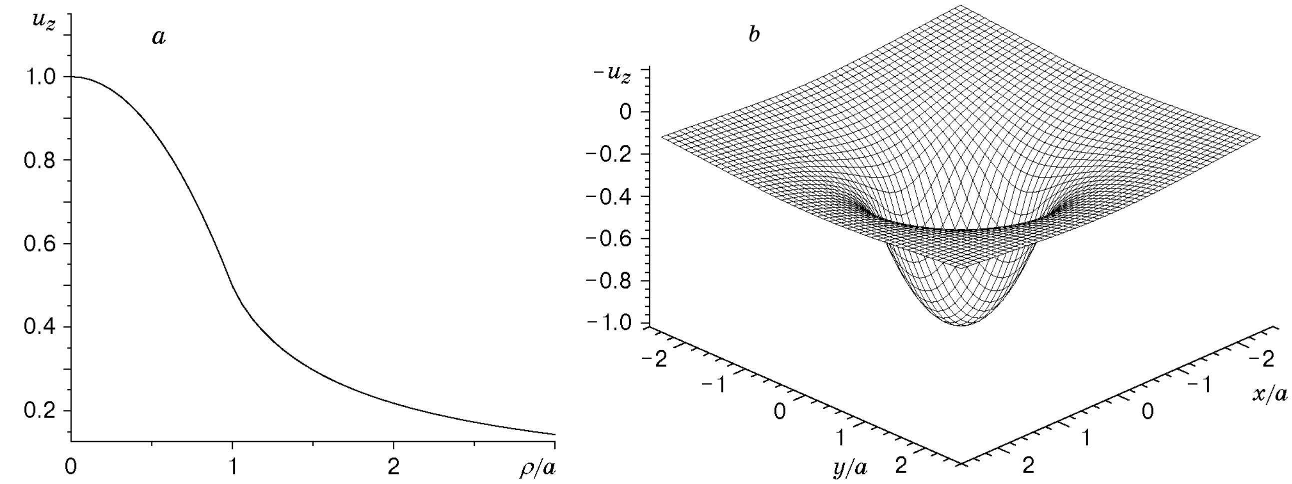

Fig. 3

In concluding, we give the expressions for the radial $[u_\rho(\rho,0)]$ and vertical $[u_z(\rho,0)]$ strain-vector components on the surface of an elastic half-space upon impression of the ball into it. Substituting (4) into (5), we obtain

$$u_\rho(\rho,0) = -\frac{N}{a}\,\frac{(1+\nu)(1-2\nu)}{2\pi E}\begin{cases}(1-(1-r^2)^{3/2})/r, & r\leqslant 1,\\ 1/r, & r\geqslant 1.\end{cases} \tag{14}$$

Hereinafter, $r \equiv \rho/a = \sqrt{(x/a)^2+(y/a)^2}$; the brace separates the coordinate-dependent part of u_ρ. Without taking into account the numerical factor in front of the brace in (14), the dependence $u_\rho(\rho,0)$ is shown in Fig. 2a. The surface $u_\rho(x,y,0)$ is shown in Fig. 2b. The radial displacement (14) is directed everywhere to the central point of tangency of the ball with the half-space surface. Its largest value is reached for $\rho = a(3/4)^{1/4} = 0.931a$, i.e., inside the contact spot, and it is 1.022 of the value of the expression $(N/a)(1+\nu)(1-2\nu)/(2\pi E)$ in (14).

The vertical displacement of the half-space surface $u_z(\rho,0)$ is an inverse Hankel transformation of expression (6) for $z=0$. Since the Hankel transformation of expression (4) (see [3, formula 6.567.1]) is

$$\bar P(w) = \int_0^a P(\rho)J_0(w\rho)\rho\,d\rho = \frac{3N}{2\pi}\int_0^1 \sqrt{1-r^2}\,J_0(awr)r\,dr = \frac{3N}{2\sqrt{2\pi}}\,\frac{J_{3/2}(aw)}{(aw)^{3/2}},$$

with allowance for (6) and (7), we have

$$u_z(\rho,0) = 2\pi\int_0^\infty \bar P(w)\bar G_z(w,0)J_0(\rho w)w\,dw = \frac{3N}{a\sqrt{2\pi}}\,\frac{1-\nu^2}{E}\int_0^\infty J_{3/2}(w_0)J_0\Big(\frac{\rho}{a}\,w_0\Big)w_0^{-3/2}\,dw_0.$$

Whence, after calculation of the integral (see [3, formula 6.574.1]), we obtain

$$u_z(\rho, 0) = \frac{3N}{4a}\,\frac{1-\nu^2}{E}\begin{cases} 1 - r^2/2, & r \leqslant 1, \\ (4/(3\pi r))F(1/2, 1/2; 5/2; 1/r^2), & r \geqslant 1, \end{cases} \tag{15}$$

where $F(\ldots)$ is a hypergeometric Gauss function. The brace in expression (15) separates the coordinate-dependent part u_z on the half-space surface. The dependence $u_z(\rho, 0)$ is shown in Fig. 3a without making allowance for the numerical factor in front of the brace in (15). The surface $u_z(x, y, 0)$ is shown in Fig. 3b. The vertical displacement of the surface (15) duplicates the form of the impressed ball in the contact spot and decreases monotonically from the maximum value $(3N/a)(1-\nu^2)/(4E)$ reached at the center of the contact spot.

REFERENCES

1. L. D. Landau and E. M. Lifshitz, *Theoretical Physics*, Vol. 7: *The Theory of Elasticity* [in Russian], Nauka, Moscow (1987).
2. K. L. Johnson, *Contact Mechanics*, Cambridge Univ. Press (1985).
3. I. I. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products*, Academic Press, New York (1980).
4. I. Sneddon, *Fourier Transforms*, McGraw-Hill, New York (1951).